

Weihrauch-completeness for layerwise computability*

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We introduce the notion of being Weihrauch-complete for layerwise computability and provide several natural examples related to complex oscillations, the law of the iterated logarithm and Birkhoff's theorem. We also consider the hitting time operators, which share the Weihrauch degree of the former examples, but fail to be layerwise computable.

1 Introduction

Layerwise computability is an effective counterpart to continuous functions that are almost-everywhere defined. This notion was introduced by HOYRUP and ROJAS [17]. A function defined on Martin-Löf random inputs is called layerwise computable, if it becomes computable if each input is equipped with some bound on the layer where it passes a fixed universal Martin-Löf test. Interesting examples of functions that are layerwise computable but not computable are obtained e.g. from Birkhoff's theorem or the study of algorithmically random Brownian motion (more below).

Weihrauch reducibility [5, 4] is a framework to compare the extent of non-computability of multivalued functions. It has been proposed with a meta-mathematical investigation of the constructive content of existence theorem in mathematics in mind. However, it has also been fruitfully employed to study (effective) function classes such as (effective) Borel measurability [3] or piecewise continuity (computability) and (effective) Δ_2^0 -measurability [33].

Our interest in this paper is in problems that are Weihrauch-complete for layerwise computability, i.e. problems that are layerwise computable, and every layerwise computable problem

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is Weihrauch reducible to it. These are, in a sense, those problems where *being layerwise computable* cannot be improved to a stronger computability notion. We shall exhibit several natural examples of problems that are Weihrauch-complete for layerwise computability.

The interaction of layerwise computability and Weihrauch reducibility has also been investigated by HÖLZL and SHAFER [16], largely in an independent development.

2 Background

We give a very brief introduction to the required concepts from randomness theory (in particular, layerwise computability) and Weihrauch reducibility. A standard reference for randomness is [27]. Layerwise computability was introduced in [17]. An extensive introduction to Weihrauch reducibility is found in the introduction of [6].

2.1 Weihrauch reducibility

We recall that a represented space $\mathbf{X} = (X, \delta_{\mathbf{X}})$ is given by a set X and a partial surjection $\delta_{\mathbf{X}} : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow X$ onto it. A partial function $F : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$ is a *realizer* of a multivalued function $f : \mathbf{X} \rightarrow \mathbf{Y}$ (in symbols $F \vdash f$), if $\delta_{\mathbf{Y}} F(p) \in f(\delta_{\mathbf{X}}(p))$ for all $p \in \text{dom}(\delta_{\mathbf{X}})$. Let $\langle \cdot, \cdot \rangle : \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$ be a standard pairing function.

Given some represented spaces \mathbf{X}, \mathbf{Y} we obtain the represented space $\mathcal{C}(\mathbf{X}, \mathbf{Y})$ of continuous functions from \mathbf{X} to \mathbf{Y} by fixing a universal oracle Type-2 machine Φ , and letting q be a name for $f : \mathbf{X} \rightarrow \mathbf{Y}$ iff $p \mapsto \Phi^q(p)$ is a realizer of f . This makes all the usual operations (in particular function application) computable. We introduce the Sierpiński-space $\mathbb{S} := (\{\top, \perp\}, \delta_{\mathbb{S}})$ where $\delta_{\mathbb{S}}(0^{\mathbb{N}}) = \perp$ and $\delta_{\mathbb{S}}(p) = \top$ if $p \neq 0^{\mathbb{N}}$. Then we can define the hyperspace $\mathcal{O}(\mathbf{X})$ of open sets by identifying a subset $U \subseteq \mathbf{X}$ with its characteristic function $\chi_U \in \mathcal{C}(\mathbf{X}, \mathbb{S})$. For the hyperspace $\mathcal{A}(\mathbf{X})$ of closed sets, we identify a subset $U \subseteq \mathbf{X}$ with the characteristic function of its complement. For details, see [32].

Two of these hyperspaces are particularly relevant for us: Regarding $\mathcal{O}(\{0, 1\}^{\mathbb{N}})$, we can envision a set $U \in \mathcal{O}(\{0, 1\}^{\mathbb{N}})$ to be given by a (finite or infinite) list of finite prefixes $(w_i)_{i \in I}$ such that $U = \bigcup_{i \in I} w_i \{0, 1\}^{\mathbb{N}}$. Regarding $\mathcal{A}(\mathbb{N})$, we can consider $A \in \mathcal{A}(\mathbb{N})$ to be given by some $p \in \mathbb{N}^{\mathbb{N}}$ such that $n \notin A \Leftrightarrow \exists i \ p(i) = n + 1$.

Now we shall introduce Weihrauch reducibility as a preorder on multivalued functions between represented spaces.

Definition 1 (Weihrauch reducibility). Let f, g be multi-valued functions on represented spaces. Then f is said to be *Weihrauch reducible* to g , in symbols $f \leq_{\text{W}} g$, if there are computable functions $K, H : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$ such that $(p \mapsto K\langle p, GH(p) \rangle) \vdash f$ for all $G \vdash g$.

The relation \leq_{W} is reflexive and transitive. We use \equiv_{W} to denote equivalence regarding \leq_{W} , and by $<_{\text{W}}$ we denote strict reducibility.

Products of represented spaces can be defined in the natural way based on $\langle \cdot, \cdot \rangle$, and we obtain products of (multivalued) functions between them accordingly. The Weihrauch degree of $f \times g$ depends only on the Weihrauch degrees of f and g , i.e. \times lifts to an operation on Weihrauch degrees as observed in [31, 5]. Besides \times , the Weihrauch degrees carry a rich algebraic structure, which we only mention in passing in Section 3, and hence refer the interested reader to [8] for the definitions.

A Weihrauch degree that is very relevant for our investigation is closed choice on the natural numbers:

Definition 2. Let $C_{\mathbb{N}} : \subseteq \mathcal{A}(\mathbb{N}) \rightrightarrows \mathbb{N}$ be defined via $n \in C_{\mathbb{N}}(A)$ iff $n \in A$.

This degree has received significant attention, e.g. in [4, 3, 28, 24, 25, 6, 29, 26]. In particular, as shown in [33], a function between computable Polish spaces is Weihrauch reducible to $C_{\mathbb{N}}$ iff it is piecewise computable iff it is effectively Δ_2^0 -measurable. For our purposes, the following representatives of the degree are also relevant:

Lemma 3. The following are Weihrauch equivalent:

1. $C_{\mathbb{N}}$
2. $UC_{\mathbb{N}}$, defined via $UC_{\mathbb{N}} = (C_{\mathbb{N}}) \upharpoonright_{\{A \in \mathcal{A}(\mathbb{N}) \mid |A|=1\}}$
3. $\min : \subseteq \mathcal{A}(\mathbb{N}) \rightarrow \mathbb{N}$
4. $\max : \subseteq \mathcal{O}(\mathbb{N}) \rightarrow \mathbb{N}$
5. $\text{Bound} : \subseteq \mathcal{O}(\mathbb{N}) \rightrightarrows \mathbb{N}$, where $n \in \text{Bound}(U)$ iff $\forall m \in U \ n \geq m$.

Proof. 1. \equiv_W 2. This is from [3].

1. \leq_W 3. Trivial.

3. \leq_W 4. Given $A \in \mathcal{A}(\mathbb{N})$, we can compute $U_{\leq A} := \{n \in \mathbb{N} \mid \forall m \in A \ n \leq m\} \in \mathcal{O}(\mathbb{N})$. Now $(\max U_{\leq A}) = \min A$.

4. \leq_W 1. If $U \in \text{dom}(\max)$, then $U \neq \emptyset$. Thus, we can assume U to be given as $U = \{p_U(n) \mid n \in \mathbb{N}\}$ for some $p \in \mathbb{N}^{\mathbb{N}}$. Now $A := \{n \in \mathbb{N} \mid \forall m \in \mathbb{N} \ p(m) \leq p(n)\}$ can be computed as a closed set. Applying $C_{\mathbb{N}}$ to A to obtain some element k , and then computing $p(k)$ yields $\max U$.

2. \leq_W 5. As before, we use $U_{\leq A}$, this time on some $A = \{n\}$. Any bound b for $U_{\leq A}$ also is a bound for n . We then simply wait until we have learned $k \notin \{n\}$ for all but one $k \leq n$ – the remaining candidate is the answer to $UC_{\mathbb{N}}$.

5. \leq_W 4. Trivial. □

We also require the following family of Weihrauch degrees:

Definition 4. Given some set $A \subseteq \mathbb{N}^{\mathbb{N}}$, let $d_A : A \rightarrow \{1\}$ be the unique map of that type.

It was shown in [15] that $d_{(\cdot)}$ is a lattice embedding of the dual of the Medvedev degrees into the Weihrauch degrees. In particular, we have that $d_A \leq_W d_B$ iff there is a computable function $F : A \rightarrow B$.

In Section 5, we also mention the degree of $\text{LPO} : \mathbb{N}^{\mathbb{N}} \rightarrow \{0, 1\}$ where $\text{LPO}(0^{\mathbb{N}}) = 1$ and $\text{LPO}(p) = 0$ for $p \neq 0^{\mathbb{N}}$ which was introduced in [38], and the Kleene star operation \star from [31, 30] defined by $f^0 := \text{id}_{\mathbb{N}^{\mathbb{N}}}$, $f^{n+1} := f^n \times f$ and $f^*(n, x) := f^n(x)$.

2.2 Randomness

Let λ denote the standard Lebesgue measure on $\{0, 1\}^{\mathbb{N}}$. A *Martin-Löf test* in $\{0, 1\}^{\mathbb{N}}$ is a computable sequence $(U_i)_{i \in \mathbb{N}}$ of open sets such that $\lambda(U_i) \leq 2^{-i}$. A Martin-Löf test $(U_i)_{i \in \mathbb{N}}$ is called *universal*, if for any Martin-Löf test $(V_i)_{i \in \mathbb{N}}$ we find that $(\bigcap_{i \in \mathbb{N}} V_i) \subseteq (\bigcap_{i \in \mathbb{N}} U_i)$. Universal Martin-Löf tests exist, and we call $\text{MLR} := \{0, 1\}^{\mathbb{N}} \setminus (\bigcap_{i \in \mathbb{N}} U_i)$ for some universal Martin-Löf

test the set of *random sequences*. The set MLR is independent of the choice of the universal test.

The informal idea behind Martin-Löf randomness is that a Martin-Löf test $(U_i)_{i \in \mathbb{N}}$ describes a very specific computable property $(\bigcap_{i \in \mathbb{N}} U_i)$, and that a random sequence should not have any very specific computable properties. Note that for any Martin-Löf test $(U_i)_{i \in \mathbb{N}}$, also $(\bigcap_{i \leq n} U_i)_{n \in \mathbb{N}}$ is a Martin-Löf test describing the same property. Thus, nothing substantial would change if we would require $U_{i+1} \subseteq U_i$ to hold in any test, i.e. would require the tests to be *nested*¹.

Following [23], a Martin-Löf test $(U_i)_{i \in \mathbb{N}}$ is called *optimal*, if for any Martin-Löf test $(V_i)_{i \in \mathbb{N}}$ we find that there is some $n \in \mathbb{N}$ such that $V_{i+n} \subseteq U_i$. Note that any optimal Martin-Löf test is necessarily universal. The existence of optimal Martin-Löf tests was established in [23].

A function $f : \text{MLR} \rightarrow \mathbf{X}$ is called *layerwise computable* (w.r.t. the universal test $(U_i)_{i \in \mathbb{N}}$), if there is a computable function $g : \subseteq \text{MLR} \times \mathbb{N} \rightarrow \mathbf{X}$ such that $p \notin U_k \Rightarrow g(p, k) = f(p)$. As shown in [16], the notion of layerwise computability does depend on the choice of universal test. If a function is layerwise computable for some universal test, then it is layerwise computable for any optimal test.

An alternate (but equivalent) approach to randomness is expressed in terms of Kolmogorov complexity. We fix a prefix-free universal Turing machine, and then let $K(w)$ be the length of the shortest programme computing the string $w \in \{0, 1\}^*$. For $p \in \{0, 1\}^{\mathbb{N}}$ and $n \in \mathbb{N}$, let $p_{\leq n}$ be the prefix of p of length n . Then for $c \in \mathbb{N}$ we set $K^d = \{p \in \{0, 1\}^{\mathbb{N}} \mid \forall n \in \mathbb{N} K(p_{\leq n}) \geq n - d\}$, and find that $\text{MLR} = \bigcup_{d \in \mathbb{N}} K^d$. Based on counting the number of prefix-free programs of a certain length, we find that $\lambda((K^d)^C) \leq 2^{-d}$; moreover, each set $(K^d)^C$ is computably open. It is known that $((K^d)^C)_{d \in \mathbb{N}}$ is a universal Martin-Löf test. For more details, see [27] for example.

3 The Weihrauch degree

Definition 5. Fix some universal Martin-Löf test $\mathcal{U} = (U_n)_{n \in \mathbb{N}}$. Let $\text{LAY}_{\mathcal{U}} : \text{MLR} \rightrightarrows \mathbb{N}$ be defined via $n \in \text{LAY}_{\mathcal{U}}(p)$ iff $p \notin U_n$. Let $\text{RD}_{\mathcal{U}} : \text{MLR} \rightarrow \mathbb{N}$ be defined via $\text{RD}_{\mathcal{U}}(p) = \min\{n \in \mathbb{N} \mid p \notin U_n\}$.

Observation 6. If $f : \text{MLR} \rightrightarrows \mathbf{X}$ is layerwise computable (w.r.t. \mathcal{U}), then $f \leq_{\mathbf{W}} \text{LAY}_{\mathcal{U}}$.

Theorem 7. $\text{LAY}_{\mathcal{U}} \equiv_{\mathbf{W}} \text{RD}_{\mathcal{U}} \equiv_{\mathbf{W}} \mathbf{C}_{\mathbb{N}} \times d_{\text{MLR}}$

Proof. $\text{LAY}_{\mathcal{U}} \leq_{\mathbf{W}} \text{RD}_{\mathcal{U}}$ Trivial.

$\text{RD}_{\mathcal{U}} \leq_{\mathbf{W}} \mathbf{C}_{\mathbb{N}} \times d_{\text{MLR}}$ As $\text{dom}(\text{RD}_{\mathcal{U}}) = \text{MLR}$, we have a random sequence available as input for d_{MLR} , and the presence of this degree does not matter further. To see that $\mathbf{C}_{\mathbb{N}}$ suffices to obtain the answer, note that given p we can compute $\{n \mid p \notin U_n\} \in \mathcal{A}(\mathbb{N})$. By Lemma 3, $\mathbf{C}_{\mathbb{N}}$ lets us compute the minimum of a closed set.

$\mathbf{C}_{\mathbb{N}} \times d_{\text{MLR}} \leq_{\mathbf{W}} \text{LAY}_{\mathcal{U}}$ By Lemma 3, we may show $\text{Bound} \times d_{\text{MLR}}$ instead. This works as follows:

The input is an enumeration of some finite set $I \subset \mathbb{N}$ (which we may safely assume to be an interval) and a random sequence p . Let w be the current prefix of the output (i.e. the

¹Which in fact was part of the original definition by MARTIN-LÖF [22]. Considering also non-nested tests though adds potential expressivity to the concept of layerwise computability, below.

input to $\text{LAY}_{\mathcal{U}}$). If we learn that $n \in I$, we consider $w0^{\mathbb{N}}$. As this is not random and \mathcal{U} is universal, we know that $w0^{\mathbb{N}} \in U_n$. As U_n is open, there is some – effectively findable – $k \in \mathbb{N}$ such that $w0^k\{0,1\}^{\mathbb{N}} \subseteq U_n$. We proceed to amend the current output to $w0^k$, and then start outputting p (until we potentially learn $n+1 \in I$).

As I is finite, the output q will have some tail identical to p , and thus is Martin Lőf random. By construction, whenever $n \in I$, then $q \in U_n$, thus if $b \in \text{LAY}_{\mathcal{U}}(q)$ then $b \in \text{Bound}(p)$. \square

There are a number of important consequences of this result. First, as the right hand side does not depend on the choice of the universal Martin Lőf test, we see that the Weihrauch degree of $\text{LAY}_{\mathcal{U}}$ and $\text{RD}_{\mathcal{U}}$ is independent of the test, too. Thus, in the following we suppress the subscript \mathcal{U} . Further consequences are:

Corollary 8. $\text{LAY} \times \text{LAY} \equiv_{\text{W}} \text{LAY}$ and $\text{LAY} \star \text{LAY} \equiv_{\text{W}} \text{LAY}$.

Proof. The former statement follows from the latter. The latter follows from Theorem 7 and $(d_{\text{MLR}} \times C_{\mathbb{N}}) \star (d_{\text{MLR}} \times C_{\mathbb{N}}) \equiv_{\text{W}} d_{\text{MLR}} \times (C_{\mathbb{N}} \star C_{\mathbb{N}})$ (as d_{MLR} is constant and $d_{\text{MLR}} \times C_{\mathbb{N}}$ is non-uniformly computable), and then $C_{\mathbb{N}} \star C_{\mathbb{N}} \equiv_{\text{W}} C_{\mathbb{N}}$ by the independent choice theorem [3]. \square

Corollary 9. $\text{LAY} <_{\text{W}} C_{\mathbb{N}}$.

Proof. As d_{MLR} is computable, we find $d_{\text{MLR}} \times C_{\mathbb{N}} \leq_{\text{W}} C_{\mathbb{N}}$. That $C_{\mathbb{N}} \not\leq_{\text{W}} d_{\text{MLR}} \times C_{\mathbb{N}}$ follows from the fact that $C_{\mathbb{N}}$ has computable inputs, whereas $d_{\text{MLR}} \times C_{\mathbb{N}}$ does not. \square

Corollary 10. $\text{LAY} \star C_{\mathbb{N}} \equiv_{\text{W}} C_{\mathbb{N}} \star \text{LAY} \equiv_{\text{W}} \text{LAY}$.

Proof. Same reasoning as for Corollary 8. \square

Corollary 11. $\text{LAY} <_{\text{W}} \widehat{\text{LAY}} \equiv_{\text{W}} \lim \times d_{\text{MLR}}$.

Proof. That $\widehat{\text{LAY}} \equiv_{\text{W}} \lim \times d_{\text{MLR}}$ follows from $\widehat{\mathbf{a} \times \mathbf{b}} \equiv_{\text{W}} \widehat{\mathbf{a}} \times \widehat{\mathbf{b}}$ as shown in [8] together with $\widehat{d_{\text{MLR}}} \equiv_{\text{W}} d_{\text{MLR}}$ and $\widehat{C_{\mathbb{N}}} \equiv_{\text{W}} \lim$ as shown in [5]. That $\lim \times d_{\text{MLR}} \not\leq_{\text{W}} \text{LAY}$ follows from LAY being non-uniformly computable and $\lim \times d_{\text{MLR}}$ being not. \square

Corollary 12. $\text{LAY} <_{\text{W}} \text{LAY}^* \equiv_{\text{W}} \text{id}_{\mathbb{N}^{\mathbb{N}}} + \text{LAY} <_{\text{W}} C_{\mathbb{N}}$.

Proof. By iterating Corollary 8 we see that $(\text{LAY})^n \equiv_{\text{W}} \text{LAY}$ for $n > 0$, this implies $\text{LAY}^* \equiv_{\text{W}} \text{id}_{\mathbb{N}^{\mathbb{N}}} + \text{LAY}$. As this degree has a computable point in its domain, we conclude $\text{LAY}^* \not\leq_{\text{W}} \text{LAY}$. As $\text{LPO} \not\leq_{\text{W}} \text{LAY}^*$, we also have $C_{\mathbb{N}} \not\leq_{\text{W}} \text{LAY}^*$. \square

Corollary 13. If $f \leq_{\text{W}} C_{\mathbb{N}}$ for $f : \subseteq \text{MLR} \Rightarrow \mathbf{Y}$, then $f \leq_{\text{W}} \text{LAY}$.

Corollary 14. The following are equivalent for $f : \subseteq \text{MLR} \rightarrow \mathbf{Y}$ for a computable metric space \mathbf{Y} :

1. f is effectively Δ_2^0 -measurable.
2. f is Π_1^0 -piecewise computable.
3. $f \leq_{\text{W}} \text{LAY}$.

Proof. This is obtained by combining the computable Jayne-Rogers theorem from [33] with Corollary 13. \square

Most results in this section were independently obtained by HÖLZL and SHAFER in [16], Corollaries 13 and 14 are inspired by their corresponding results though. The proofs in [16] differ significantly from ours, in particular, they give direct proofs of the claims listed as corollaries here.

In a very similar fashion to Theorem 7, we can also characterize the degree of Kolmogorov randomness. While this technically is just a special case of Theorem 7, we provide a direct proof in the hope to illuminate the underlying phenomena. Let $\text{Kol} : \text{MLR} \rightarrow \mathbb{N}$ be defined via $\text{Kol}(p) := \min\{c \in \mathbb{N} \mid \forall n \in \mathbb{N} K(p_{\leq n}) \geq n - c\}$. Then:

Proposition 15. $\text{Kol} \equiv_W C_{\mathbb{N}} \times d_{\text{MLR}}$

Proof. Note that $\{c \in \mathbb{N} \mid \forall n \in \mathbb{N} K(p_{\leq n}) \geq n - c\}$ can be computed as a closed set from p – if some c is not in that set, we can find some n and some short program (of length less than $n - c$) producing the prefix $p_{\leq n}$. The reduction $\text{Kol} \leq_W C_{\mathbb{N}} \times d_{\text{MLR}}$ then follows from Lemma 3.

For the other direction, we show $\text{Bound} \times d_{\text{MLR}} \leq_W \text{Kol}$ and again invoke Lemma 3. Given some $w \in \{0, 1\}^n$ and $k \in \mathbb{N}$, there will be some programme for our fixed universal machine printing $w0^k$ of size $O(n \log k)$. Based on the constant involved, n and c we can choose k sufficiently large such that $K(w0^k) + c < n + k$.

Now our reduction works as follows: Copy the random sequence serving as the input to d_{MLR} over to the input for Kol. Whenever we learn that some c is in the input to Bound, we pick a k based on the current prefix of the input to Kol and c and write the corresponding number of zeros. Then we continue to copy the random sequence. Eventually the input to Bound stabilizes, so our input for Kol will actually be random. Moreover, by constructing, the output of Kol will exceed all numbers in the input to Bound. \square

4 Examples of Weihrauch-complete layerwise computable operations

4.1 Complex oscillations

Let $\mathcal{C}_0([0, 1], \mathbb{R})$ denote the space of continuous functions $f : [0, 1] \rightarrow \mathbb{R}$ where $f(0) = 0$. The complex oscillations CO (introduced in [1]) are the Martin-Löf random elements (in the sense of [19]) of $\mathcal{C}_0([0, 1], \mathbb{R})$ equipped with the Wiener measure. They are of great interest as generic representatives of Brownian motion [12]. We shall consider a specific bijection $\Phi : \text{MLR} \rightarrow \text{OC}$ studied in [12].

The definition of Φ is as follows:

$$\Phi(\alpha)(t) = g(\alpha_0)\Delta_0(t) + g(\alpha_1)\Delta_1(t) + \sum_{j \geq 1} \sum_{n < 2^j} g(\alpha_{jn})\Delta_{jn}(t). \quad (4)$$

The $\Delta_0(t), \Delta_1(t), \Delta_{jn}(t)$ are the sawtooth functions obtained by integrating from 0 to t the elements of the Haar system of functions $e_0 = 1, e_1 = \chi([0, \frac{1}{2})) - \chi([\frac{1}{2}, 1))$, $e_{jn} = \{\chi([n2^{-j}, n2^{-j} + 2^{-(j+1)})) - \chi([n2^{-j} + 2^{-(j+1)}, (n+1)2^{-j}))\}2^{j/2}$, $0 \leq n < 2^j$ and $j \geq 1$.

The numbers $\alpha_0, \alpha_1, \alpha_{jn}$ are subsequences of α and the $g(\alpha)$ are then defined via the normal distribution $\alpha = \int_{-\infty}^{g(\alpha)} \frac{e^{-t^2/2}}{\sqrt{2\pi}} dt$, $\alpha \in (0, 1)$. Define, for each k , the set $B_k := (\alpha_0, \alpha_1, \alpha_{jn} : 1 \leq j \leq k, 0 \leq n < 2^j)$, and set $L_k = 2^{k+1}$. This is the number of elements in the sequence B_k . Thus B_k consists of the first 2^{k+1} α 's.

We define the (elements of the) array B_k from α in stages as follows: At stage one we use the first $4L_1 = 4(2^2) = 16$ bits of α to obtain the first 4 bits of each of $\alpha_0, \alpha_1, \alpha_{10}, \alpha_{11}$. In general we use $\bar{\alpha}(4kL_k)$, that is the initial segment of α of length $4kL_k$, to (computably) fill the first $4k$ positions of all elements of B_k .

We now set $\eta_0 = g(\alpha_0)$, $\eta_1 = g(\alpha_1)$ and $\eta_{jn} = g(\alpha_{jn})$. These are independent $N(0, 1)$ random variables w.r.t. the Lebesgue measure.

We shall require the basic:

Fact 16. There is a computable function $\eta : \text{MLR} \rightarrow \mathbb{R}$ inducing the normal distribution $\mathcal{N}(0, 1)$ on \mathbb{R} .

Observation 17. $\max : \mathcal{C}([0, 1], \mathbb{R}) \rightarrow \mathbb{R}$ and $\text{GreaterNat} : \mathbb{R} \rightrightarrows \mathbb{N}$ where $n \in \text{GreaterNat}(x)$ if $x \leq n$ are computable.

Lemma 18. [12] The function $\Phi : \text{MLR} \rightarrow \text{CO}$ can be recursively recursively defined from the values $\Phi(\alpha)$ takes on the dyadic rationals, and then extending it continuously to the interval. To wit:

1. $\Phi(\alpha)(1) := \eta(\alpha_0)$
2. $\Phi(\alpha)(\frac{1}{2}) := \frac{1}{2} (\eta(\alpha_0) + \eta(\alpha_1))$
3. $\Phi(\alpha)(\frac{2n+1}{2^{j+1}}) := \frac{1}{2} (2^{-j/2} \eta(\alpha_{jn}) + \Phi(\alpha)(\frac{n+1}{2^j}) + \Phi(\alpha)(\frac{n}{2^j}))$

Lemma 19. Given $k \in \mathbb{N}$ and $v \in \{0, 1\}^*$ we can compute some $w \in \{0, 1\}^*$ such that for all $\alpha \in \text{MLR}$ we find that $k < \sup_{t \in [0, 1]} \Phi(vw\alpha)(t)$.

Proof. Pick some $j, n \in \mathbb{N}$ such that α_{jn} in $\alpha = \langle \alpha_0, \alpha_1, \dots, \alpha_{jn}, \dots \rangle$ does not depend on the prefix of length $|v|$ at all. We can then choose a prefix of α_{jn} (and prefixes of the $\alpha_{j'n'}$) to enforce that $\eta(\alpha_{jn})$ is large enough to ensure that $\Phi(\beta)(\frac{2n+1}{2^{j+1}}) > k$ for all β sharing these prefixes. From these prefixes, we obtain w . \square

Theorem 20. $\Phi \equiv_W \text{LAY}$.

Proof. It was shown in [11] that Φ is layerwise computable. We sketch the argument: From the definition of $\Phi(\alpha)$ we learn how to compute the values taken by $\Phi(\alpha)$ on a dense subset. To obtain $\Phi(\alpha)$ as an element in $\mathcal{C}([0, 1], \mathbb{R})$, we also need a modulus of convergence. In [13], it is shown that the following holds for sufficiently small h :

$$\sup_{t \in [0, 1]} |\Phi(\alpha)(t+h) - \Phi(\alpha)(t)| < \sqrt{3h \log(h^{-1})}$$

By inspecting we see that knowing a bound for the layer of α suffices to determine what *small enough* means for h – and then we have a modulus of convergence.

That $\Phi \leq_W \text{LAY}$ then follows immediately by Observation 6, so it only remains for us to show $\text{LAY} \leq_W \Phi$. By Theorem 7 and Lemma 3, we can show $d_{\text{MLR}} \times \text{Bound} \leq_W \Phi$ instead. We start to use the Martin-Löf random α obtained as input to d_{MLR} as input to Φ . Whenever we find some k in the input to Bound while the current prefix of the input to Φ is v , we extend by w as in Lemma 19, and then continue to write α . As the input to Bound will stabilize, this procedure produces some $\beta \in \text{MLR}$. Moreover, we find that if $K \in \text{GreaterNat}(\max(\Phi(\beta)))$, then K is a valid output for Bound . \square

4.2 Law of the iterated logarithm

The *law of the iterated logarithm* states that a one-dimensional random walk will eventually remain within a given sublinear (in time) bounds around the origin. We consider its effective version:

Definition 21. Let $\text{LIL} : \text{MLR} \rightrightarrows \mathbb{N}$ be defined via $N \in \text{LIL}(\alpha)$ iff:

$$\forall n \geq N \quad \left| \sum_{i=0}^{n-1} (2\alpha(i) - 1) \right| < \sqrt{2n \log \log n}$$

It was shown by VOVK [36] that LIL is well-defined, and inspection of the proof also yields that LIL is layerwise computable.

Lemma 22. Given $N \in \mathbb{N}$ and $u \in \{0, 1\}^*$ we can compute some $v \in \{0, 1\}^*$ such that $|uv| > N$ and $\left| \sum_{i=0}^{|uv|-1} (2(uv)(i) - 1) \right| > \sqrt{2|uv| \log \log |uv|}$.

Proof. Let $|u| = k$, and assume v is of the form $v = 1^{k+l}$ for some $l \in \mathbb{N}$. Then $\left| \sum_{i=0}^{|uv|-1} (2(uv)(i) - 1) \right| \geq l$. Thus choosing $l > N - k$ satisfying $l > \sqrt{2(2k + l) \log \log (2k + l)}$ suffices for our purpose. This in turn can be achieved by $l \geq \max\{20, 2k\}$. \square

Theorem 23. $\text{LIL} \equiv_{\text{W}} \text{LAY}$.

Proof. The direction $\text{LIL} \leq_{\text{W}} \text{LAY}$ follows from Observation 6 and VOVK's result [36]. For the other direction, we show $d_{\text{MLR}} \times \text{Bound} \leq_{\text{W}} \text{LIL}$ instead and employ Theorem 7 and Lemma 3.

The random input to d_{MLR} is copied to the input to LIL. If a new number N appears in the input to Bound while the current prefix to the input for LIL is v , we extend the input to LIL according to Lemma 22. Then we continue to copy over the random input. As the input to Bound will stabilize eventually, this procedure results in a random input to LIL, and by constructing, any output from LIL will be a valid output for Bound. \square

4.3 Birkhoff's theorem

The convergence speed in a special case of Birkhoff's theorem was one of the first examples of a layerwise-computable map, already given as such in [18, Theorem 5.2.4] based on earlier work by the third author [10] and by GALATOLO, HOYRUP and ROJAS [14]. Here we shall only consider a toy version – which already is Weihrauch-complete for layerwise computability (which then of course is inherited by any more general but still layerwise computable versions).

Let $S : \{0, 1\}^{\mathbb{N}} \rightarrow \{0, 1\}^{\mathbb{N}}$ be the usual shift-operator, and $\pi_1 : \{0, 1\}^{\mathbb{N}} \rightarrow \{0, 1\}$ be the projection to the first bit. Let Birkhoff : $\text{MLR} \times \mathbb{N} \rightrightarrows \mathbb{N}$ be defined via $N \in \text{Birkhoff}(p, k)$ iff $\forall n \geq N$ we find that:

$$\left| \left(\frac{1}{n+1} \sum_{i=0}^n \pi_1(S^i(p)) \right) - \frac{1}{2} \right| < 2^{-k}$$

Lemma 24. Given $u \in \{0, 1\}^*$ and $k, N \in \mathbb{N}$, $k > 0$, we can compute some $v \in \{0, 1\}^*$ such that $|uv| \geq N$ and:

$$\left| \left(\frac{1}{|uv|} \sum_{i=0}^{|uv|-1} \pi_1(S^i(uv)) \right) - \frac{1}{2} \right| > 2^{-k}$$

Proof. Choosing $v := 0^l$ for sufficiently large l makes the statement true, and we can decide for any value of l whether it is already large enough. \square

Theorem 25. $\text{Birkhoff} \equiv_W \text{LAY}$

Proof. The reduction $\text{Birkhoff} \leq_W \text{LAY}$ follows from results from [10] and Observation 6.

For the reverse direction, we show $d_{\text{MLR}} \times \text{Bound} \leq_W \text{Birkhoff}$ instead, invoking Theorem 7 and Lemma 3. We copy the random sequence provided as input to d_{MLR} over to the input for Birkhoff. If some number N is listed in the input to Bound, we extend the current input to Birkhoff as in Lemma 24 with w as the current prefix of the input to Birkhoff and $k = 1$. After that, we proceed to copy the random sequence.

Eventually, the input to Bound stabilizes, so the input p to Birkhoff has a random tail and thus is random itself. By construction, if $N \in \text{Birkhoff}(p, 1)$, then N is a valid output for Bound. \square

Effective versions of Birkhoff's theorem for sets that are either effectively open or effectively closed have been found [2, 20]. In general however, the rate of convergence is not layerwise computable². The lower bound for the Weihrauch degree of finding such a rate of convergence provided in Theorem 25 of course still applies, but finding upper bounds and a precise classification seems to be an interesting open area.

4.4 Random harmonic series

The harmonic series $\sum_{n \in \mathbb{N}} \frac{1}{n}$ might be the most famous example of a diverging series. If, however, the signs of the summands are chosen by independent coin flips, the resulting series will almost-surely converge. Some observations on the resulting distribution can be found in [35]. The effective counterpart was found by DAI:

Theorem 26 ([9, Theorem 2], Special case). The map $p \mapsto \sum_{n \in \mathbb{N}} \frac{(-1)^{p(n)}}{n} : \text{MLR} \rightarrow \mathbb{R}$ is well-defined and layerwise computable.

Theorem 27. $\left(p \mapsto \sum_{n \in \mathbb{N}} \frac{(-1)^{p(n)}}{n} : \text{MLR} \rightarrow \mathbb{R} \right) \equiv_W \text{LAY}$

Proof. The reduction from left to right follows from [9, Theorem 2] and Observation 6. By Theorem 7 and Lemma 3 we can show $d_{\text{MLR}} \times \text{Bound} \leq_W \left(p \mapsto \sum_{n \in \mathbb{N}} \frac{(-1)^{p(n)}}{n} : \text{MLR} \rightarrow \mathbb{R} \right)$ for the other direction.

Given some $p \in \text{MLR}$ and an increasing bounded sequence $(a_i)_{i \in \mathbb{N}}$, we will obtain some $q \in \text{MLR}$ by changing finitely many 0's to 1's in p such that we can guarantee $\sum_{n \in \mathbb{N}} \frac{(-1)^{q(n)}}{n} \geq \max_{n \in \mathbb{N}} a_n$. For this, we inspect both the sequence $(a_n)_{n \in \mathbb{N}}$ and compute the partial sums $\sum_{n=0}^N \frac{(-1)^{q(n)}}{n}$ for the output written so far. If for some $N \in \mathbb{N}$ we find that $\sum_{n=0}^N \frac{(-1)^{q(n)}}{n} < a_N$, then we identify finitely many $j_k > \dots > j_0 > N$ with $p(j_i) = 0$ and $\sum_{n=0}^N \frac{(-1)^{q(n)}}{n} + 2 \sum_{l=0}^k \frac{1}{j_l} > a_N + 1$. Such j_l must exist as the harmonic series diverges. We then let define $q(m)$ for $N < m \leq j_k$ by $q(m) = 1$ if $m = j_l$ for some l and $q(m) = p(m)$ else.

²A counterexample had already been presented in [37, Theorem 1]. While the result is formulated in terms of non-effectiveness of convergence in probability, it is easily seen that layerwise computability of the (pointwise) rate of convergence implies effective convergence in probability.

Let $c = \sum_{n \in \mathbb{N}} \frac{(-1)^{p(n)}}{n}$, let t be such that $|\sum_{n=t}^{\infty} \frac{(-1)^{p(n)}}{n}| < 1$ and $N \geq \max_{n \in \mathbb{N}} a_n$. If the procedure above is triggered k times, then $\sum_{n \in \mathbb{N}} \frac{(-1)^{q(n)}}{n} \geq c+k$ is ensured, as each time the limit is increased by at least 1. Once $c+k \geq N+1$, and we have processed p up to at least position t , it follows that the procedure cannot be triggered again. Thus, the Hamming distance of p and q is finite, and hence $q \in \text{MLR}$ follows. That the limit satisfies the criterion is immediate. \square

In [9], a general result was established regarding when some limit of the form $\sum_{n \in \mathbb{N}} (-1)^{p(n)} a_n$ is guaranteed to exist for $p \in \text{MLR}$, and moreover, to be layerwise computable. We point out that the proof of Theorem 27 is not referring to specific properties of the harmonic series beyond its divergence, and hence extends in a straight-forward manner to a more general case.

As a consequence of Theorem 27, we can find an example for a problem that is layerwise computable, not computable and not Weihrauch complete for layerwise computability. This example was suggested as a promising candidate to the authors by Mathieu Hoyrup and Laurent Bienvenu at CCR 2015.

Corollary 28. The map $\text{SumApr} : \text{MLR} \times \mathbb{Q} \times \mathbb{N} \rightrightarrows \{0,1\}$ with $0 \in \text{SumApr}(p, q, k)$ if $\sum_{n \in \mathbb{N}} \frac{(-1)^{p(n)}}{n} < q + 2^{-k}$ and $1 \in \text{SumApr}(p, q, k)$ if $\sum_{n \in \mathbb{N}} \frac{(-1)^{p(n)}}{n} > q$ is

1. layerwise computable,
2. not computable,
3. not Weihrauch complete for layerwise computability.

Proof. 1. As a consequence of Theorem 26.

2. If SumApr were computable, then we could compute the map from Theorem 27 by exhaustive search, contradicting that theorem.
3. It was shown in [28] that $\text{C}_{\mathbb{N}}$ is not reducible to any map with finite range even relative to some oracle. As $d_{\text{MLR}} \times \text{C}_{\mathbb{N}}$ is equivalent to $\text{C}_{\mathbb{N}}$ relative to any ML-random oracle, the claim follows from Theorem 7. \square

The generalization from random harmonic series to random Fourier series was explored by POTGIETER [34], and might provide for further examples of problems that are Weihrauch-complete for layerwise computability.

4.5 A digression on strong Weihrauch reducibility

In some situations a more restricted version of Weihrauch reducibility, namely strong Weihrauch reducibility, is relevant. While the structure of the strong Weihrauch degrees is less well-behaved than the standard version, but there are some interesting operations preserving only strong Weihrauch degrees, but not Weihrauch degrees (cf. [7]).

Definition 29 (Weihrauch reducibility). Let f, g be multi-valued functions on represented spaces. Then f is said to be *strongly Weihrauch reducible* to g , in symbols $f \leq_{\text{sw}} g$, if there are computable functions $K, H : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$ such that $KGH \vdash f$ for all $G \vdash g$.

Observation 30. Let $f : \text{MLR} \rightrightarrows \mathbb{N}$ be such that $f(p)$ can be computed from the layer of p only (without access to p). Then $f \leq_{\text{sw}} \text{LAY}$.

Corollary 31. $\text{LIL} \leq_{\text{sw}} \text{LAY}$

Corollary 32. Birkhoff \leq_{sW} LAY

On the other hand, if a function is strongly Weihrauch reducible to LAY, then it can only take countably many different values. Thus we find:

Observation 33. $\Phi \not\leq_{\text{sW}}$ LAY

Observation 34. $\left(p \mapsto \sum_{n \in \mathbb{N}} \frac{(-1)^{p(n)}}{n} : \text{MLR} \rightarrow \mathbb{R}\right) \not\leq_{\text{sW}}$ LAY

5 Hitting time

Natural counterexamples³ to Observation 6 (i.e. problems that are Weihrauch reducible to LAY but not layerwise computable) are found in the hitting time operators. These take an additional input besides the random sequence though, and we need to clarify what layerwise computability could mean here. Two approaches make sense: A function $f : \text{MLR} \times \mathbf{X} \rightarrow \mathbf{Y}$ shall be called *weakly layerwise computable*, if for every computable $x \in \mathbf{X}$ the slice function $f_x : \text{MLR} \rightarrow \mathbf{Y}$ with $f_x(p) = f(p, x)$ is layerwise computable. We call it *strongly layerwise computable* if for any n the restriction $f|_{U_n^C \times \mathbf{X}}$ is computable.

By $\mathcal{O}^+(\{0, 1\}^{\mathbb{N}})$ we denote the restriction of $\mathcal{O}(\{0, 1\}^{\mathbb{N}})$ to non-empty sets. Let $\mathcal{A}_{\lambda > 0}(\{0, 1\}^{\mathbb{N}})$ be the restriction of $\mathcal{A}(\{0, 1\}^{\mathbb{N}})$ to sets of positive Lebesgue measure. Let $T : \{0, 1\}^{\mathbb{N}} \rightarrow \{0, 1\}^{\mathbb{N}}$ be the usual shift-operator, and let $\text{HittingTime}_{\mathcal{O}} : \text{MLR} \times \mathcal{O}^+(\{0, 1\}^{\mathbb{N}}) \rightarrow \mathbb{N}$ and $\text{HittingTime}_{\mathcal{A}} : \text{MLR} \times \mathcal{A}_{\lambda > 0}(\{0, 1\}^{\mathbb{N}}) \rightarrow \mathbb{N}$ be defined via $\text{HittingTime}(p, U) = \min\{n \in \mathbb{N} \mid T^n(p) \in U\}$. That these maps are well-defined is a classic result by KUČERA [21].

Theorem 35. $\text{HittingTime}_{\mathcal{O}} \equiv_{\text{W}} d_{\text{MLR}} \times \text{LPO}^* <_{\text{W}}$ Layer, but $\text{HittingTime}_{\mathcal{O}}$ is not strongly⁴ layerwise computable.

Proof. 1. $\text{HittingTime}_{\mathcal{O}} \leq_{\text{W}} d_{\text{MLR}} \times \text{LPO}^*$

Note that $(\min : \subseteq \mathcal{O}(\mathbb{N}) \rightarrow \mathbb{N}) \equiv_{\text{W}} \text{LPO}^*$. Given $p \in \{0, 1\}^{\mathbb{N}}$, $U \in \mathcal{O}(\{0, 1\}^{\mathbb{N}})$ we can compute $\{n \mid T^n(p) \in U\} \in \mathcal{O}(\mathbb{N})$. The claim follows.

2. $d_{\text{MLR}} \times \text{LPO}^* \leq_{\text{W}} \text{HittingTime}_{\mathcal{O}}$ Again, we use $(\min : \subseteq \mathcal{O}(\mathbb{N}) \rightarrow \mathbb{N}) \equiv_{\text{W}} \text{LPO}^*$ and show $d_{\text{MLR}} \times \min \leq_{\text{W}} \text{HittingTime}_{\mathcal{O}}$ instead. Our input is some $p \in \text{MLR}$ and some non-empty $U \in \mathcal{O}(\mathbb{N})$. We inspect U until we found some element $b \in U$ (which provides an upper bound for $\min U$).

We proceed to construct the random sequence q used as the first input to $\text{HittingTime}_{\mathcal{O}}$. For $i \leq b$, let $w_i \in \{0, 1\}^{2+2\lceil \log b \rceil}$ be the sequence that starts with 11 and then intersperses zeros and the digits in a binary code for i of length $\lceil \log b \rceil$. Then we let $q := w_0 w_1 \dots w_b p$. Next, we construct the open set $V \in \mathcal{O}(\{0, 1\}^{\mathbb{N}})$ used as the second input to $\text{HittingTime}_{\mathcal{O}}$. We let $V = \bigcup_{\{i \leq b \mid i \in U\}} w_i \{0, 1\}^{\mathbb{N}}$. Then we find that for $j \leq (2 + 2\lceil \log b \rceil)b$ we have $T^j(q) \in V$ iff $j = l(2 + 2\lceil \log b \rceil)$ and $l \in U$. As we can compute l from j and b , the reduction works.

3. $\text{HittingTime}_{\mathcal{O}}$ is not strongly layerwise computable.

Fix any $p \in \text{MLR}$. The map $U \mapsto \text{HittingTime}_{\mathcal{O}}(p, U)$ is Weihrauch equivalent to LPO – in particular no restriction on the first component is computable. □

³The existence of counterexamples, albeit of a more technical nature, is also shown in [16].

⁴We do not know whether this map could be weakly layerwise computable at the present.

Theorem 36. $\text{HittingTime}_{\mathcal{A}} \equiv_W \text{Layer}$, but $\text{HittingTime}_{\mathcal{A}}$ is not weakly layerwise computable.

Proof. 1. $\text{HittingTime}_{\mathcal{A}} \leq_W \text{Layer}$

By definition, every instance (p, A) to $\text{HittingTime}_{\mathcal{A}}$ computes a Martin-Löf random p . It only remains to prove that $\text{HittingTime}_{\mathcal{A}}$ is computable with finitely many mindchanges. Start by proclaiming that 0 is a valid answer, until $p \notin A$ is confirmed (which will happen, iff true). Then claim that 1 is a valid answer, until $tp \notin A$ is confirmed, etc.

2. $\text{Layer} \leq_W \text{HittingTime}_{\mathcal{A}}$

Instead, we show $d_{\text{MLR}} \times \text{Bound} \leq_W \text{HittingTime}_{\mathcal{A}}$ and use Lemma 3. Starting with $p \in \text{MLR}$ and some $q \in \mathbb{N}^{\mathbb{N}}$ s.t. $\exists N \in \mathbb{N} \{0, \dots, N\} = \{q(i) \mid i \in \mathbb{N}\}$, we wish to compute some $A \in \mathcal{A}_{\lambda > 0}(\{0, 1\}^{\mathbb{N}})$ s.t. $\forall i \in \mathbb{N} \exists t \in \mathbb{N} p_{[i, t]} \notin A$. Given $p \in \{0, 1\}^{\mathbb{N}}$ and $i < j \in \mathbb{N}$, let $p_{[i, j]} \in \{0, 1\}^{j-i}$ denote the subword of p from position i to position j . Now, simply set $A = \left(\bigcup_{i \in \mathbb{N}} p_{[q(i), 2q(i)+1]} \{0, 1\}^{\mathbb{N}} \right)^C$. This is a closed set computable from p and q , and by construction satisfies our criterion. It only remains to show that the set has positive measure, which follows from the fact that $\sum_{i \in I} 2^{-i-1} < 1$ for any finite set $I \subset \mathbb{N}$.

3. $\text{HittingTime}_{\mathcal{A}}$ is not weakly layerwise computable.

For $p \in \{0, 1\}^{\mathbb{N}}$ and $s, t \in \mathbb{N}$, let $p_{[s, t]} \in \{0, 1\}^{\mathbb{N}}$ be defined via $p_{[s, t]}(n) = p(n)$ for $n < s$ or $n > s + t$ and $p_{[s, t]}(n) = 0$ otherwise. Fix some $p \in K^1$. We claim that there is some $B \in \mathbb{N}$ and a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that for any $n \in \mathbb{N}$ we find that $p_{[n, f(n)]} \in K^B$ and $p_{[n, f(n)]} \notin K^1$. The (presumably non-computable) function f can be constructed by choosing $f(n)$ minimal for the second condition.

Now let us assume that $\text{HittingTime}_{\mathcal{A}}$ is called on input K^1 and q where $q = p$ or $q = p_{[n, f(n)]}$. By assumption, B is a valid output for the layer of q . But the output needs to be 0 if $q = p$, and some $t > 0$ if $q = p_{[n, f(n)]}$. But as $\lim_{n \rightarrow \infty} p_{[n, f(n)]} = p$, this case distinction cannot be made in a computable way. □

Complementing the results above, the map $\text{HittingTime}_{\mathcal{A} \wedge \mathcal{O}}$ where the set-input is demanded to be clopen is easily seen to be computable.

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